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# THE RELATIVE SENSITIVITIES OF PRE- AND POST-DETECTION INTEGRATORS

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# THE RELATIVE SENSITIVITIES OF PRE- AND POST-DETECTION INTEGRATORS*

## INTRODUCTION

When a signal is known to lie somewhere within a given band of frequencies, there are two means available for increasing the probability of detection of that signal for a given false-alarm rate. The first method, called pre-detection, or IF integration, involves the use of a bank of narrow-band filters connected in parallel and staggered in frequency so as to cover the entire band of possible frequencies. Each of these filters then feeds its own biased-diode alarm circuit.

In the second method, called post-detection, or video integration, the band in which the signal is known to lie is isolated by a bandpass filter, detected to beat the unknown frequency down to zero frequency; the signal is then passed through a low-pass filter, the cutoff frequency of which will usually be equal to the half-width of the narrow-band filters used in the other method. These widths will be determined by the possible frequency spread of the signal. Schematic diagrams for these two systems are shown in Fig. 1.

If it is necessary to retain information as to the frequency of the signal, obviously one must use pre-detection integration; but if this information is not required, then the choice between the two systems must be based on a consideration of complexity and sensitivity. We shall compare the two systems by determining the input signal-to-noise ratios required to give the same per cent detectabilities and the same false-alarm rates. Since the results are practically independent of the choice of false-alarm probabilities in the range $10^{-5} < P_N < 10^{-14}$ and per cent detectabilities in the range $50\% < P_s < 98\%$ we shall make our calculations assuming an alarm bias level sufficient to give a false-alarm probability of $\sim 10^{-5}$ and a signal-to-noise ratio at the alarm circuit sufficient to give a 50 per cent probability of detection.

S/N at the input to alarm circuits:- The signal out of the low-pass filter in the post-detection system will approximate a Gaussian distribution provided the reduction in bandwidth by the low-pass filter is considerable (by a factor > 10). Thus, by setting the alarm level four standard deviations above the mean noise voltage, we shall realize a false-alarm probability of $P_N \sim 3 \times 10^{-5}$.

To match this performance in the pre-detection system, the false-alarm probability for each of the n alarms must be given by $P_N \sim (3 \times 10^{-5})/n$. The probability distribution for the envelope of a sine wave in narrow-band noise is given by Rice** as

$$P\left(\frac{R}{\sigma}\right) \frac{dR}{\sigma} = \frac{R}{\sigma 2} \exp\left[-(R^2 + P^2)/2\sigma^2\right] I_o\left(\frac{RP}{\sigma 2}\right) dR \quad , \tag{1}$$

where $\sigma^2$ is the mean-square noise voltage, P is the amplitude of the signal, and $I_o$ is the modified Bessel function of the first kind and zero order. The moments of the distribution are given by

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*This paper contains results previously given by other authors. The simplicity of the present approach is the only attribute of the presentation. Cf. Sperry Report No. 5223-1109; RAND, RM-753; D. Middleton, Proc. IRE 36, 1467 (1948).

**S. O. Rice, Bell Sys. Tech. Jour. 25, 151 (1945).

$$\overline{R^n} = (2\sigma^2)^{n/2}\,\Gamma\,(1 + n/2)\;{}_1F_1\,(-n/2; 1; -\Sigma_o) \quad , \tag{2}$$

in which $\Sigma_o = P^2/2\sigma^2$ and is thus the output signal-to-noise power ratio. If the level of an envelope detector is adjusted to $V_o$ volts, the false-alarm probability will be given by

$$P_N = \int_{V_o}^{\infty} P(R)\, dR = \int_{V_o}^{\infty} \frac{R}{\sigma^2} \exp[-R^2/2\sigma^2]\, dR = \exp[-V_o^2/2\sigma^2] \quad . \tag{3}$$

Thus we may make a table of alarm levels as a function of the number of filters.

TABLE I

Alarm Levels for Pre-Detection Integration

| n | 1 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|
| $V_o^2/2\sigma^2$ | 10.3* | 13.8 | 14.5 | 15.2 | 15.9 | 16.6 |

The signal-to-noise power ratio $\Sigma_o$ required to give 50 per cent detection probability may be found by choosing $\Sigma_o$ such that the first moment of the envelope distribution will be equal to $V_o$. By using the asymptotic expansion of the confluent hypergeometric function

$$ {}_1F_1(-k; 1; -X) \approx \frac{X^k}{\Gamma(k+1)} \left[1 + \frac{k^2}{1!X} + \frac{k^2(K-1)^2}{2!X^2} + \ldots\right] \quad , \tag{4}$$

we obtain

$$V_o = \sqrt{2\sigma^2 \Sigma_o} \quad \left[1 + \frac{1}{4\Sigma_o} + \ldots\right]$$

or

$$\Sigma_o \simeq V_o^2/2\sigma^2 \quad . \tag{5}$$

<u>S/N at input to narrow-band filters</u>:- It is now easy to deduce the required signal-to noise-ratios $\Sigma_1$ at the input to the pre-detection system. If we assume n equal rectangular narrow-band filters to cover the full band, then $\Sigma_1 = \Sigma_o/n$.

For the practical case in which LCR (simple, single-tuned) filters are used, the input signal-to-noise ratios would have to be 3 db greater than those given in Table II for rectangular filters. (Noise bandwidth = $\pi/2 \times 3$ db band width.) This is because of the increased noise passed by the optical filter (a factor of $\pi/2$) and the small average decrease in signal power due to the fact that the signal may not fall at the center of a filter (a factor of $4/\pi$).

*The alarm level for one filter was included in order to show the small effect on sensitivity that results from knowing in which filter the signal will appear. If this information is known in a system consisting of 32 filters, all but one of the filters can be turned off and the alarm level lowered by only 1.3 db to realize the same false-alarm probability.

TABLE II

$\Sigma_1$ vs n for Pre-Detection Integration

| n | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|
| $\Sigma_1$ (rect) db | −3.7 | −6.4 | −9.2 | −12.1 | −15 |
| $\Sigma_1$ (LRC) db | −0.7 | −3.4 | −6.2 | − 9.1 | −12 |

The spectral distribution from a full-wave square-law detector:- In order to determine $\Sigma_1$ for the post-detector integration system, we must analyze the action of the detector in some detail. It can be shown* in a rigorous fashion that, if a rectangular band of white noise constitutes the input to a full-wave square-law detector, the output spectrum is as shown in Fig. 2, where the low-frequency components consist of a DC component plus a triangular distribution of low-frequency components. We shall present a rather naive derivation of these low-frequency components in order to bring out more clearly the physics of the problem.

A pure sine wave will be converted into a sine-squared wave by the detector and hence will contribute a DC component of amplitude equal to $a^2/2$ where a is the amplitude of the original sine wave. Now we may consider an infinitesimally narrow band of the noise spectrum in which the power is given by Adf. The amplitude of the wave, which in the limit is a pure sine wave, is $\sqrt{2Adf}$ and this is converted by the detector into a sine-squared wave of amplitude 2Adf. The DC voltage contributed by the entire band of noise is thus (2AB/2) and the DC power is $A^2B^2$. This DC component will not interest us directly, but it will help make clear the conditions under which the output of the rectifier plus low-pass filter will be essentially Gaussian. We shall designate this term (NXN).

The low-frequency components are given by the mixing of two different components of noise and we shall designate such terms by (NXN'). If two frequency components $X(t) = a \sin \omega_1 t + b \sin \omega_2 t$ are passed through a square-law detector, we get the output

$$y(t) = X^2(t) = \frac{a^2 + b^2}{2} + ab \cos(\omega_1 - \omega_2)t - \left(\frac{a^2}{2} \cos 2\omega_1 t + \frac{b^2}{2} \cos 2\omega_2 t\right.$$
$$\left. + ab \cos(\omega_1 + \omega_2)t\right) \qquad (6)$$

The first term in this expression gives the DC contribution from (NXN) already discussed. The second term gives the low-frequency components from (NXN'). Thus the power contributed to the noise in a frequency interval between f and (f + Δf) by the mixing between an infinitesimal band of width df' and a finite band of width Δf separated by a frequency f is proportional to

$$(\sqrt{2A\Delta f} \cdot \sqrt{2Adf'})^2 = 2A^2 \Delta f \, df' .$$

The power is now summed over f' holding f and Δf constant, giving

$$W(f)\Delta f = \int_f^B 2A^2 \Delta f \, df' = 2A^2(B-f)\Delta f \quad (f \le B) , \qquad (7)$$

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*W. B. Davenport, Jr., unpublished notes on Noise Theory.

where W(f) is the power density in the detected signal.

Since the power in the low-frequency rectified noise band is as great as the power in the rectified DC noise component while the detected signal can never go negative, the unfiltered output of the detector cannot possibly be Gaussian. However, if we filter out all but the very lowest frequencies, the power in the low-frequency rectified noise that is passed by the filter can be made much smaller than the DC component and, further, is almost white. Such an argument makes it appear plausible, at least, that the noise now approximates a Gaussian distribution. More rigorous treatments prove that such is the case.

Input S/N for the post-detection system:- If we add a pure sine-wave signal to the noise, we have two more terms to add to the output. The signal can beat against itself to give a DC signal term (SXS) in the output power proportional to $P^4/4$. The signal can also beat against the noise to give additional noise (SXN), and the amount of this contribution will depend on how near one edge of the broad frequency band the signal lies. Let the distance in frequency from the signal to the nearer edge of the band be designated by $f_o$. For frequencies in the output less than $f_o$, the power due to (SXN) is proportional to $(P^2 A \Delta f)$; for frequencies in the range $(f_o < f < B - f_o)$, the contribution is $(1/2\ P^2 A \Delta f)$; while for higher frequencies, it is zero.

If we now pass the output through a square low-pass filter which passes frequencies out to B/2n we have the following contributions to the noise power:

$$(NXN') \propto A^2 B^2/n \quad ,$$
$$(SXN) \propto P^2 AB/2n \quad .$$

The last term is correct unless the signal lies within B/2n of the edge of the band, a situation that we shall ignore.

The signal-to-noise relations are somewhat confused in the post-detection system by the fact that the noise at the output of the detector is increased when a signal is introduced. Let us designate the mean voltage and standard deviation in the absence of signal by $\overline{m}(0)$ and $\sigma(0)$, respectively. The same quantities with signal present we shall designate by $\overline{m}(X)$ and $\sigma(X)$. The signal voltage we shall designate by $[\overline{m}(X) - \overline{m}(0)]$.

If, in the absence of signal, the alarm level $(V_o)$ is set four standard deviations above the mean noise level so that $V_o = 4\sigma(0)$, the false-alarm probability will be given by $P_N \sim 3 \times 10^{-5}$. If signal is now introduced, the mean voltage and standard deviation will both increase. For the general case, the input signal-to-noise ratio required to give a certain per cent detection probability would be dependent on (SXS), (NXN') and (SXN). However, for the special case of 50 per cent detection probability, the criterion is simply that the mean voltage in the presence of signal must increase to the alarm level. Thus we have that

$$\overline{m}(X) - \overline{m}(0) = V_o = 4\sigma(0).$$

We see that the criterion for 50 per cent detection probability can be given in terms of the ratio of signal power to zero signal noise power $(\Sigma'_o)$ as

$$\Sigma'_o = \frac{V_o^2}{2\sigma^2(o)} = 8$$

and hence is not dependent on the noise contributed by (SXN) terms.

$\Sigma'_o$ can be expressed in turn by

$$\Sigma'_o = \frac{(SXS)}{(NXN')} = \frac{P^4}{4} \cdot \frac{n}{A^2B^2} = n\Sigma_1^{\ 2} \text{ (rect. filter)} \tag{8}$$

for rectangular filters of width B/2n. We now can form a table of $\Sigma_1$ vs n for the case of a post-detection system using a full-wave square-law detector.

TABLE III

$\Sigma_1$ vs n for a Post-Detection System

| n | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|
| $\Sigma_1$ (rect) db | –3 | –4.5 | –6 | –7.5 | –9 |
| $\Sigma_1$ (RC) db | –1.0 | –2.5 | –4.0 | –5.5 | –7.0 |

The loss due to the use of an RC filter in this case only involves the additional noise passed by the filter since the signal always appears at zero frequency.

Thus for the RC filters we have

$$\Sigma'_o = \frac{2n}{\pi} \Sigma_1^{\ 2} \text{ (RC filter)} \quad , \tag{9}$$

where the half-power point of the RC filter falls at a frequency given by B/2n.

A comparison between Tables III and II will now show the relative sensitivities of pre- and post-detection systems. The results of the comparison depend upon the particular problem. For example, let us assume a signal can be expected to last a given length of time. This time will determine the optimum width for the single filter in the post-detection system. If the target can be expected to have no radial component of acceleration, then n will be the same for the pre-detection as for the post-detection system, and a comparison of input signal-to-noise ratios for the same value of n is called for. One then concludes that the pre-detection system is better by from 0 to 5 db for the range of n included in the tables. However, if the target is accelerated, it may spend appreciably less than full time in any given filter in the pre-detection system. In addition, propeller modulation and variations in aspect could conceivably broaden the spectrum more than scanning. Hence the sensitivity in the pre-detection system in practical cases could drop to the point where the two systems are essentially equivalent.

In some cases, the optimum number of filters is so large and the expectation of accelerated targets sufficiently high that one may use less than the optimum number of filters and not sacrifice much in the average sensitivity. In such cases, however, the post-detection system can still be made optimum, and hence the comparison will be for different values of n in the two cases. Thus if one uses 32 filters in the pre-detection system but can use n = 128 in the post-detection system, the post-detection system is better than the pre-detection system by 3 db. At least some, and maybe all, of this loss could be regained by using a hybrid system that consisted of the pre-detection filter system followed by a post-detector integrator in each channel to give an over-all integration time equal to the time on-target.

Effect of detector law:- The above calculations have been based on the full-wave square law detector. Middleton* has shown that the sensitivity of systems using half-wave square-law detectors is identical with that given for full-wave square-law detectors, and the half-wave linear detector is never better or worse by as much as 0.2 db. Thus we may conclude that the exact nature of the detector is not important in the present considerations.

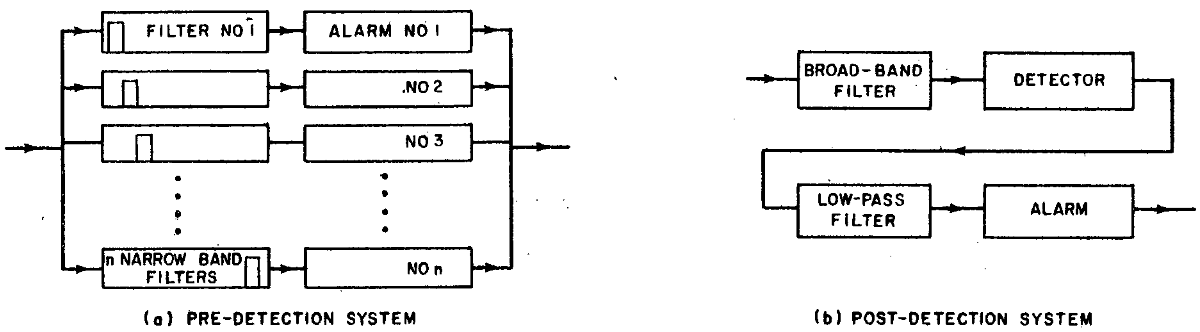


Fig. 1. Block diagrams for the two systems to be compared.

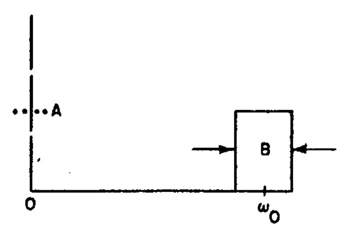


Fig. 2(a). Input spectrum into full wave square law detector.

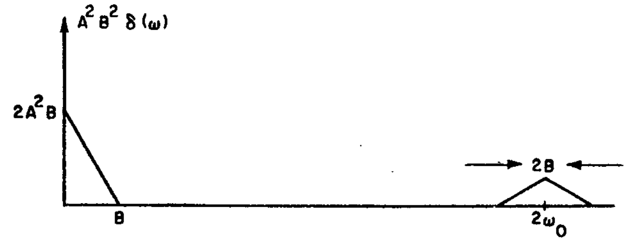


Fig. 2(b). Output spectrum from full wave square law detector.

*D. Middleton, Proc. IRE 36, 1467 (1948).